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Short Communication

Approximate solution of a strongly nonlinear complex differential equation

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1. Introduction

The vibrations of a one-mass system with two degrees of freedom are mostly described using a second-order differential equation with a complex dependent variable. The differential equation is usually linear as is shown in the papers of Dimentberg [1] and Vance [2]. The solution of the differential equation clarifies the linear phenomena which occur in the system. If in the system some small nonlinearities exist they are introduced in the differential equation of motion as small nonlinear terms. In papers [3–5] the various methods for solving differential equations with complex dependent variable and small nonlinearity are introduced. The solutions obtained describe the influence of small nonlinearities on the behavior of the system. As is known, in the real system both weak and also strong nonlinearities act. The motion of the system is described by a second-order strongly nonlinear complex differential equation. Some special cases of such differential equations are considered. In [6] the one-frequency solution of a special type of Duffing equation is obtained. Besides the Duffing type of nonlinearity [7], the Liénard and Rayleigh systems with complex functions are considered in [8]. The interaction between strong and weak nonlinearity in a system with complex dependent variable is also discussed in [9]. An approximate analytic solution procedure is developed for analyzing such a system. The main disadvantage of the suggested procedures is that they do not give the general solution but are convenient only for some special cases of nonlinearities and corresponding special initial conditions.

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In this paper the initial conditions are arbitrary but there is a constraint to the differential equation: the coupling of the differential equation is due to the small nonlinearity. Separating the strong nonlinear term in the differential equation with complex dependent variable into real and imaginary parts leads to functions that depend only on one real function and its time derivative. The real part depends only on a function $x(t)$ and its time derivative $\dot{x}(t)$ and the imaginary part on a function $y(t)$ and its time derivative $\dot{y}(t)$. The mathematical model of the system is

$$\ddot{z} + f(z, \dot{z}) = \varepsilon\phi(z, \dot{z}, z^*, \dot{z}^*), \tag{1}$$

where $z = x + \mathrm{i}y$ is a complex function, $z^* = x - \mathrm{i}y$ is complex conjugate, $\mathrm{i} = \sqrt{-1}$ is the imaginary unit, x and y are real functions of time t, $\dot{z} = \mathrm{d}z/\mathrm{d}t$ is the first time derivative of the complex function z, $\dot{z}^* = \mathrm{d}z^*/\mathrm{d}t$ is the first time derivative of the complex conjugate function z^*, $\ddot{z} = \mathrm{d}^2z/\mathrm{d}t^2$ is the second time derivative of the complex function z, $\varepsilon \ll 1$ is a small parameter, $\phi = \phi_1 + \mathrm{i}\phi_2$ is the small nonlinear function, and

$$f(z, \dot{z}) = f_1 + \mathrm{i}f_2 \tag{2}$$

with

$$f_1 \equiv f_1(z + z^*, \dot{z} + \dot{z}^*) \equiv f_1(x, \dot{x}),$$
$$f_2 \equiv f_2(\mathrm{i}(z - z^*), \mathrm{i}(\dot{z} + \dot{z}^*)) \equiv f_2(y, \dot{y}).$$

The arbitrary initial conditions are

$$z(0) = z_0, \quad \dot{z}(0) = \dot{z}_0. \tag{3}$$

In this paper an approximate analytic solution procedure based on perturbation of the generating solution is developed. First, the closed form analytic solution of two independent single-degree-of-freedom systems, which are two decoupled strongly nonlinear second-order differential equations ($\varepsilon = 0$ in Eq. (1)), is developed. The trial solution in the form of generating solution is formed and the differential equation of motion (1) is transformed into the system of four first-order differential equations. The solution of this system of differential equations represents the solution of Eq. (1). The suggested procedure is applied to system with strong cubic nonlinearity of Duffing type. The two-frequency solution is the Jacobi elliptic function. For the general case, an averaging procedure for solving such differential equations with small nonlinearity is developed. The method is used for calculation of the vibration properties of a rotor system with pure cubic nonlinearity and small nonlinearity of van der Pol type.

2. Generating solution

For the case when the small nonlinearity is neglected and $\varepsilon = 0$, the differential equation (1) transforms to

$$\ddot{z} + f(z, \dot{z}) = 0, \tag{4}$$

where $f(z, \dot{z}) = f_1(x, \dot{x}) + \mathrm{i}f_2(y, \dot{y})$. It is a strongly nonlinear differential equation. By separating the real and imaginary parts of the differential equation (4) and substituting $z = x + \mathrm{i}y$ into (3) the following two independent differential equations are obtained:

$$\ddot{x} + f_1(x, \dot{x}) = 0, \quad x(0) = x_0, \quad \dot{x}(0) = \dot{x}_0, \tag{5}$$

$$\ddot{y} + f_2(y, \dot{y}) = 0, \quad y(0) = y_0, \quad \dot{y}(0) = \dot{y}_0. \tag{6}$$

The solutions of the equations are independent, and have the form

$$x = x(t, A, \alpha), \quad y = y(t, B, \beta), \tag{7}$$

where t is time, (A, α) are parameters which depend on the initial conditions (5), and (B, β) are parameters which depend on the initial conditions (6). Specifically, A and B are the initial amplitudes, and α and β are the initial phase angles. In spite of the fact that (4) is a strongly nonlinear differential equation, the generating solution is a superposition of solutions (7), i.e.,

$$z = x(t, A, \alpha) + \mathrm{i}y(t, B, \beta). \tag{8}$$

It represents the closed form solution of Eq. (4) which satisfies constraint (2).

3. Trial solution

Based on the generating solution the trial solution is formed. The following constraints are introduced:

1. The trial solution has the form of the generating solution and it is

$$z = x(t, A(t), \alpha(t)) + \mathrm{i}y(t, B(t), \beta(t)), \tag{9}$$

   where $A(t), \alpha(t)$ and $B(t), \beta(t)$ are time variable functions. Solution (9) has to satisfy the differential equation (1) with limitation (2).
2. The first time derivative of (9) has the form of the first time derivative of the generating solution (8) where A, α, B, and β are supposed to have constant values

$$\dot{z} = \frac{\partial x(t, A(t), \alpha(t))}{\partial t} + \mathrm{i}\,\frac{\partial y(t, B(t), \beta(t))}{\partial t}. \tag{10}$$

   The additional terms which exist due to the fact that $A(t), \alpha(t)$ and $B(t), \beta(t)$ are time dependent give us a new relation

$$\left(\dot{A}\,\frac{\partial x}{\partial A} + \dot{\alpha}\,\frac{\partial x}{\partial \alpha}\right) + \mathrm{i}\left(\dot{B}\,\frac{\partial y}{\partial B} + \dot{\beta}\,\frac{\partial y}{\partial \beta}\right) = 0. \tag{11}$$

3. The time derivative of (10) is

$$\ddot{z} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial x(t, A(t), \alpha(t))}{\partial t} + \mathrm{i}\frac{\partial y(t, B(t), \beta(t))}{\partial t}\right). \tag{12}$$

Substituting solution (9) the corresponding time derivatives (10) and (12) into (1) lead to

$$\left(\dot{A}\,\frac{\partial}{\partial A}+\dot{\alpha}\,\frac{\partial}{\partial \alpha}\right)\left(\frac{\partial x}{\partial t}\right)+\mathrm{i}\left(\dot{B}\,\frac{\partial}{\partial B}+\dot{\beta}\,\frac{\partial}{\partial \beta}\right)\left(\frac{\partial y}{\partial t}\right)=\varepsilon\phi_1+\mathrm{i}\varepsilon\phi_2, \tag{13}$$

where $\phi_p \equiv \phi_p(x, y, \partial x/\partial t, \partial y/\partial t), p = 1, 2$.

On separating the real and imaginary parts in (11) and (13) and after some transformation, the following four first-order differential equations are obtained:

$$\begin{aligned}
\dot{A} &= \varepsilon\,\mathrm{Re}(\phi_1+\mathrm{i}\phi_2)/\left(\left(\frac{\partial}{\partial A}-\left(\frac{\partial x}{\partial A}\Big/\frac{\partial x}{\partial \alpha}\right)\frac{\partial}{\partial \alpha}\right)\left(\frac{\partial x}{\partial t}\right)\right),\\
\dot{\alpha} &= -\varepsilon\,\mathrm{Re}(\phi_1+\mathrm{i}\phi_2)/\left(\left(\left(\frac{\partial x}{\partial \alpha}\Big/\frac{\partial x}{\partial A}\right)\frac{\partial}{\partial A}-\frac{\partial}{\partial \alpha}\right)\left(\frac{\partial x}{\partial t}\right)\right),\\
\dot{B} &= \varepsilon\,\mathrm{Im}(\phi_1+\mathrm{i}\phi_2)/\left(\left(\frac{\partial}{\partial B}-\left(\frac{\partial y}{\partial B}\Big/\frac{\partial y}{\partial \beta}\right)\frac{\partial}{\partial \beta}\right)\left(\frac{\partial y}{\partial t}\right)\right),\\
\dot{\beta} &= -\varepsilon\,\mathrm{Im}(\phi_1+\mathrm{i}\phi_2)/\left(\left(\left(\frac{\partial y}{\partial \alpha}\Big/\frac{\partial y}{\partial B}\right)\frac{\partial}{\partial B}-\frac{\partial}{\partial \beta}\right)\left(\frac{\partial y}{\partial t}\right)\right).
\end{aligned} \tag{14}$$

By solving Eqs. (14) the functions $A(t), \alpha(t), B(t), \beta(t)$ are determined, i.e., the exact solution (9). Unfortunately, obtaining the closed form solution of system (14) is usually impossible. Some approximation has to be introduced. As the motion is periodic it means that the solutions are also periodic functions. At this point it is convenient to introduce the averaging of Eqs. (14). Solutions of the averaged first-order differential equations represent the approximate solution of the differential equation (1).

4. Strongly nonlinear Duffing equation

Let us consider a special case of the differential equation (1) where the strong nonlinearity is of Duffing type. The differential equation is

$$\ddot{z}+b_1 z+b_3\left(\left(\frac{z+z^*}{2}\right)^3-\left(\frac{z-z^*}{2}\right)^3\right)=\varepsilon(\phi_1+\mathrm{i}\phi)_2. \tag{15}$$

The generating solution of the generating equation

$$\ddot{z}+b_1 z+b_3\left(\left(\frac{z+z^*}{2}\right)^3-\left(\frac{z-z^*}{2}\right)^3\right)=0 \tag{16}$$

is

$$z = A\,\mathrm{cn}(\omega_1 t+\alpha, m_1)+\mathrm{i}B\,\mathrm{cn}(\omega_2 t+\beta, m_2), \tag{17}$$

where cn is the Jacobi elliptic function [10] with frequency ω_1 and ω_2, and modulus m_1 and m_2, respectively. Substituting (17) into (16) and separating the real and imaginary part we obtain

$$\omega_1^2 = b_1 + b_3A^2, \quad m_1 = \frac{b_3A^2}{2\omega_1^2},$$
$$\omega_2^2 = b_1 + b_3B^2, \quad m_2 = \frac{b_3B^2}{2\omega_2^2}, \tag{18}$$

where due to the initial values the arbitrary parameters A, B, α, and β satisfy the relations

$$x_0 = A\,\mathrm{cn}\left(\alpha, \frac{b_3A^2}{2(b_1+b_3A^2)}\right), \quad y_0 = B\,\mathrm{cn}\left(\beta, \frac{b_3B^2}{2(b_1+b_3B^2)}\right),$$
$$\dot{x}_0 = -A\sqrt{b_1+b_3A^2}\,\mathrm{sn}\left(\alpha, \frac{b_3A^2}{2(b_1+b_3A^2)}\right)\mathrm{dn}\left(\alpha, \frac{b_3A^2}{2(b_1+b_3A^2)}\right),$$
$$\dot{y}_0 = -B\sqrt{b_1+b_3B^2}\,\mathrm{sn}\left(\beta, \frac{b_3B^2}{2(b_1+b_3B^2)}\right)\mathrm{dn}\left(\beta, \frac{b_3B^2}{2(b_1+b_3B^2)}\right). \tag{19}$$

The functions sn and dn are also Jacobi elliptic functions [10].

The trial solution is according to (17)

$$z = A(t)\mathrm{cn}(\psi_1, m_1) + \mathrm{i}B(t)\mathrm{cn}(\psi_2, m_2) \equiv A\,\mathrm{cn}_1 + \mathrm{i}B\,\mathrm{cn}_2, \tag{20}$$

where $\mathrm{cn}_1 \equiv \mathrm{cn}(\psi_1, m_1)$, $\mathrm{cn}_2 \equiv \mathrm{cn}(\psi_2, m_2)$, $\psi_1 = \int_0^t \omega_1\,\mathrm{d}t + \alpha(t)$, $m_1 \equiv m_1(t)$, $\psi_2 = \int_0^t \omega_2\,\mathrm{d}t + \beta(t)$, $m_2 \equiv m_2(t)$. Applying the suggested procedure the four first-order differential equations which correspond to the differential equation (15) are

$$\dot{A} = \frac{\varepsilon\,\mathrm{cn}_{1\psi}\,\mathrm{Re}(\phi_1 + \mathrm{i}\phi_2)}{\mathrm{cn}_{1\psi}^2(\omega_1 + A\omega_1') + A\omega_1 m_1'\,\mathrm{cn}_{1\psi}\,\mathrm{cn}_{1\psi m} - \omega_1\mathrm{cn}_{1\psi\psi}(\mathrm{cn}_1 + Am_1'\,\mathrm{cn}_{1m})},$$
$$A\dot{\alpha} = \frac{\varepsilon(\mathrm{cn}_1 + Am_1'\,\mathrm{cn}_{1m})\,\mathrm{Re}(\phi_1 + \mathrm{i}\phi_2)}{\mathrm{cn}_{1\psi}^2(\omega_1 + A\omega_1') + A\omega_1 m_1'\,\mathrm{cn}_{1\psi}\,\mathrm{cn}_{1\psi m} - \omega_1\,\mathrm{cn}_{1\psi\psi}(\mathrm{cn}_1 + Am_1'\,\mathrm{cn}_{1m})},$$
$$\dot{B} = \frac{\varepsilon\,\mathrm{cn}_{2\psi}\,\mathrm{Im}(\phi_1 + \mathrm{i}\phi_2)}{\mathrm{cn}_{2\psi}^2(\omega_2 + B\omega_1') + B\omega_2 m_2'\,\mathrm{cn}_{2\psi}\,\mathrm{cn}_{2\psi m} - \omega_2\,\mathrm{cn}_{2\psi\psi}(\mathrm{cn}_2 + Bm_2'\,\mathrm{cn}_{2m})},$$
$$B\dot{\beta} = \frac{\varepsilon(\mathrm{cn}_2 + Am_2'\,\mathrm{cn}_{2m})\,\mathrm{Im}(\phi_1 + \mathrm{i}\phi_2)}{\mathrm{cn}_{2\psi}^2(\omega_2 + B\omega_1') + B\omega_2 m_2'\,\mathrm{cn}_{2\psi}\,\mathrm{cn}_{2\psi m} - \omega_2\,\mathrm{cn}_{2\psi\psi}(\mathrm{cn}_2 + Bm_2'\,\mathrm{cn}_{2m})}, \tag{21}$$

where $(\cdot)' \equiv \partial/\partial A$, $(\mathrm{cn})_\psi = \partial(\mathrm{cn})/\partial\psi$ is the derivative with respect to the argument, and $(\mathrm{cn})_m = \partial(\mathrm{cn})/\partial m$ is the derivative with respect to the modulus.

The averaged differential equations are

$$\dot{A} = \left\langle \frac{\varepsilon\,\mathrm{cn}_{1\psi}\,\mathrm{Re}(\phi_1 + \mathrm{i}\phi_2)}{\mathrm{cn}_{1\psi}^2(\omega_1 + A\omega_1') + A\omega_1 m_1'\,\mathrm{cn}_{1\psi}\,\mathrm{cn}_{1\psi m} - \omega_1\,\mathrm{cn}_{1\psi\psi}(\mathrm{cn}_1 + Am_1'\,\mathrm{cn}_{1m})} \right\rangle,$$
$$A\dot{\alpha} = -\left\langle \frac{\varepsilon(\mathrm{cn}_1 + Am_1'\,\mathrm{cn}_{1m})\,\mathrm{Re}(\phi_1 + \mathrm{i}\phi_2)}{\mathrm{cn}_{1\psi}^2(\omega_1 + A\omega_1') + A\omega_1 m_1'\,\mathrm{cn}_{1\psi}\,\mathrm{cn}_{1\psi m} - \omega_1\,\mathrm{cn}_{1\psi\psi}(\mathrm{cn}_1 + Am_1'\,\mathrm{cn}_{1m})} \right\rangle,$$
$$\dot{B} = \left\langle \frac{\varepsilon\,\mathrm{cn}_{2\psi}\,\mathrm{Im}(\phi_1 + \mathrm{i}\phi_2)}{\mathrm{cn}_{2\psi}^2(\omega_2 + B\omega_1') + B\omega_2 m_2'\,\mathrm{cn}_{2\psi}\,\mathrm{cn}_{2\psi m} - \omega_2\,\mathrm{cn}_{2\psi\psi}(\mathrm{cn}_2 + Bm_2'\,\mathrm{cn}_{2m})} \right\rangle,$$
$$B\dot{\beta} = -\left\langle \frac{\varepsilon(\mathrm{cn}_2 + Am_2'\,\mathrm{cn}_{2m})\,\mathrm{Im}(\phi_1 + \mathrm{i}\phi_2)}{\mathrm{cn}_{2\psi}^2(\omega_2 + B\omega_1') + B\omega_2 m_2'\,\mathrm{cn}_{2\psi}\,\mathrm{cn}_{2\psi m} - \omega_2\,\mathrm{cn}_{2\psi\psi}(\mathrm{cn}_2 + Bm_2'\,\mathrm{cn}_{2m})} \right\rangle, \tag{22}$$

where $\langle \cdot \rangle = \int_0^{4K_1(m_1)} \int_0^{4K_2(m_2)} (\cdot)\,\mathrm{d}\psi_1\,\mathrm{d}\psi_2$ and $K_1(m_1)$ and $K_2(m_2)$ are the complete elliptic integrals of the first kind [10].

For the special case when the nonlinearity is of the pure cubic type ($b_1 = 0$), the modulus of the Jacobi elliptic functions is constant and has the value $m_1 = m_2 = \frac{1}{2}$. The frequency parameter is $\omega_1 = A\sqrt{b_3}$ and $\omega_2 = B\sqrt{b_3}$, respectively. Then, the first-order differential equations (22) are simplified to

$$\dot{A} = \frac{\varepsilon\,\mathrm{cn}_{1\psi}\,\mathrm{Re}(\phi_1 + \mathrm{i}\phi_2)}{A\sqrt{b_3}(2\mathrm{cn}_{1\psi}^2 - \mathrm{cn}_{1\psi\psi}\,\mathrm{cn}_1)},$$
$$A\dot{\alpha} = \frac{\varepsilon\,\mathrm{cn}_1\,\mathrm{Re}(\phi_1 + \mathrm{i}\phi_2)}{A\sqrt{b_3}(2\mathrm{cn}_{1\psi}^2 - \mathrm{cn}_{1\psi\psi}\,\mathrm{cn}_1)},$$
$$\dot{B} = \frac{\varepsilon\,\mathrm{cn}_{2\psi}\,\mathrm{Im}(\phi_1 + \mathrm{i}\phi_2)}{B\sqrt{b_3}(2\mathrm{cn}_{2\psi}^2 - \mathrm{cn}_{2\psi\psi}\,\mathrm{cn}_2)},$$
$$B\dot{\beta} = -\frac{\varepsilon\,\mathrm{cn}_2\,\mathrm{Im}(\phi_1 + \mathrm{i}\phi_2)}{B\sqrt{b_3}(2\mathrm{cn}_{2\psi}^2 - \mathrm{cn}_{2\psi\psi}\,\mathrm{cn}_2)}, \tag{23}$$

where the arguments are $\psi_1 = \int_0^t A\sqrt{b_3}\,\mathrm{d}t + \alpha(t)$ and $\psi_2 = \int_0^t B\sqrt{b_3}\,\mathrm{d}t + \beta(t)$.

When the strong nonlinearity is zero ($b_3 = 0$) we obtain $\omega_1 = \omega_2 = \omega = \sqrt{b_1}$, a value which is independent of the initial conditions, and $m_1 = m_2 = 0$. The elliptic function cn transforms to a circular function (cos). The four first-order differential equations (21) transform to

$$\dot{A} = -\varepsilon \sin\psi_1\,\mathrm{Re}(\phi_1 + \mathrm{i}\phi_2), \quad A\dot{\alpha} = \varepsilon\cos\psi_1\,\mathrm{Re}(\phi_1 + \mathrm{i}\phi_2),$$
$$\dot{B} = -\varepsilon \sin\psi_2\,\mathrm{Im}(\phi_1 + \mathrm{i}\phi_2), \quad B\dot{\beta} = \varepsilon\cos\psi_2\,\mathrm{Im}(\phi_1 + \mathrm{i}\phi_2), \tag{24}$$

where $\psi_1 = \int_0^t \omega\,\mathrm{d}t + \alpha(t)$, $\psi_2 = \int_0^t \omega\,\mathrm{d}t + \beta(t)$, and ϕ_1 and ϕ_2 are functions of $x = A\cos\psi_1$, $y = B\cos\psi_2$, $\dot{x} = -\omega A\sin\psi_1$, and $\dot{y} = -\omega B\sin\psi_2$. Introducing the averaging procedure the following four first-order differential equations are obtained:

$$\dot{A} = -\varepsilon\langle \sin\psi_1\,\mathrm{Re}(\phi_1 + \mathrm{i}\phi_2)\rangle, \quad A\dot{\alpha} = \varepsilon\langle \cos\psi_1\,\mathrm{Re}(\phi_1 + \mathrm{i}\phi_2)\rangle,$$
$$\dot{B} = -\varepsilon\langle \sin\psi_2\,\mathrm{Im}(\phi_1 + \mathrm{i}\phi_2)\rangle, \quad B\dot{\beta} = \varepsilon\langle \cos\psi_2\,\mathrm{Im}(\phi_1 + \mathrm{i}\phi_2)\rangle, \tag{25}$$

where $\langle \cdot \rangle \equiv \int_0^{2\pi} \int_0^{2\pi} (\cdot)\,\mathrm{d}\psi_1\,\mathrm{d}\psi_2$.

5. Example: small nonlinearity of van der Pol type

Let us consider an example when the strong nonlinearity is of pure cubic type and the small nonlinearity is of van der Pol type. The mathematical model is

$$\ddot{z} + b_3\left(\left(\frac{z+z^*}{2}\right)^3 - \left(\frac{z-z^*}{2}\right)^3\right) = \varepsilon\dot{z}[1 - p(zz^*)], \tag{26}$$

where p is a constant parameter. According to (23), the first-order differential equations of motion are

$$\dot{A} = \varepsilon A(1 - pA\,\mathrm{cn}_1^2 - pB\,\mathrm{cn}_2^2)\,\mathrm{sn}_1^2\,\mathrm{dn}_1^2, \tag{27}$$

$$A\dot{\alpha} = \varepsilon\,\mathrm{sn}_1\,\mathrm{cn}_1\,\mathrm{dn}_1\,A(1 - pA\,\mathrm{cn}_1^2 - pB\,\mathrm{cn}_2^2), \tag{28}$$

$$\dot{B} = \dot{\varepsilon}B(1 - pA\,\mathrm{cn}_1^2 - pB\,\mathrm{cn}_2^2)\,\mathrm{sn}_2^2\,\mathrm{dn}_2^2, \tag{29}$$

$$B\dot{\beta} = \varepsilon\,\mathrm{sn}_2\,\mathrm{cn}_2\,\mathrm{dn}_2\,B(1 - pA\,\mathrm{cn}_1^2 - pB\,\mathrm{cn}_2^2). \tag{30}$$

Averaging Eqs. (28) and (30) and integrating them for the initial conditions

$$\alpha(0) = \alpha_0, \quad \beta(0) = \beta_0, \tag{31}$$

the solutions are $\alpha = \alpha_0 = \text{const.}$, $\beta = \beta_0 = \text{const.}$

Averaging the Eqs. (27) and (29) the following coupled differential equations are obtained:

$$\frac{\dot{A}}{\varepsilon A} = a - pbA^2 + pacB^2, \quad \frac{\dot{B}}{\varepsilon B} = a - pbB^2 + pacA^2, \tag{32}$$

where $a = \langle \mathrm{sn}_1^2\,\mathrm{dn}_1^2 \rangle = \langle \mathrm{sn}_2^2\,\mathrm{dn}_2^2 \rangle$, $b = \langle \mathrm{sn}_1^2\,\mathrm{cn}_1^2\,\mathrm{dn}_1^2 \rangle = \langle \mathrm{sn}_2^2\,\mathrm{cn}_2^2\,\mathrm{dn}_2^2 \rangle$, $c = \langle \mathrm{cn}_1^2 \rangle = \langle \mathrm{cn}_2^2 \rangle$. Eliminating the variable B in the system of differential equations (32) the following differential equation is obtained:

$$\ddot{A} + \dot{A}^2 f(A) + \varepsilon\dot{A}g(A) + \varepsilon^2 h(A) = 0, \tag{33}$$

where

$$f(A) = -\frac{1}{A}\left(1 - \frac{2b}{ac}\right), \quad g(A) = -2\left(\frac{2b}{c} + a\right) + 2pA^2\left(b + \frac{2b^2}{ac} - ac\right),$$
$$h(A) = 2A^2\left[a(a - pbA^2) + pA^2\left(a^2c - ab - \frac{2b}{c}\right) - p^2A^4 b\left(a - \frac{b^2}{ac}\right)\right]. \tag{34}$$

The differential equation (33) is a second-order Abel equation. Neglecting the ε^2 terms of second-order small value ε^2 and introducing the new function $y(A) = \dot{A}(t)$ into (33) leads to the first-order differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}A} + f(A)y = -\varepsilon g(A), \tag{35}$$

i.e.,

$$\dot{A} = \varepsilon \frac{a}{b}(2b + ac)A - \varepsilon p A^3 \frac{(2b + ac)b - a^2c^2}{b + ac}. \tag{36}$$

Integrating relation (36) for the initial condition $A(0) = A_0$, we have

$$\frac{A}{A_0}\sqrt{\frac{pa_2A_0^2 - a_1}{pa_2A^2 - a_1}} = \exp(\varepsilon a_1 t), \tag{37}$$

i.e.,

$$A = \frac{A_0}{\sqrt{(1 - (pa_2A_0^2/a_1))\exp(-2\varepsilon a_1 t) + (pa_2A_0^2/a_1)}}, \tag{38}$$

where

$$a_1 = \frac{a(2b - ac)}{b}, \quad a_2 = \frac{(2b - ac)b + a^2c^2}{b - ac}. \tag{39}$$

Eliminating the variable A in (32) and using the previously assumed calculating procedure, we obtain

$$B = \frac{B_0}{\sqrt{(1 - (pa_2B_0^2/a_1))\exp(-2\varepsilon a_1 t) + (pa_2B_0^2/a_1)}}. \tag{40}$$

To present the correctness of the suggested analytical procedure a numerical example is considered. Let us assume that the parameters in the differential equation (26) are $b_3 = 1$, $p = 1$ and $\varepsilon = 0.01$. For the initial conditions $A(0) = A_0 = 0.1$, $B(0) = B_0 = 0.2$, $\alpha(0) = \alpha_0 = \pi/3$, and $\beta(0) = \beta_0 = \pi/6$ using relations (31), (38), and (40) the analytical solutions $x_A - t$ and $y_A - t$ are plotted in Fig. 1. In the same figure the numerical solutions $x_N - t$ and $y_N - t$ of Eq. (26) obtained by applying the Runge–Kutta procedure are plotted. The solutions are compared. It is evident that the difference between the analytical and numerical solutions is negligible for small parameter values.

Analyzing Eqs. (38) and (40) it is concluded that

(1) for $(1 - (pa_2A_0^2/a_1)) > 0$ and $(1 - (pa_2B_0^2/a_1)) > 0$ the amplitudes A and B increase to a limit value $A = B = \sqrt{a_1/pa_2}$.
(2) for $(1 - (pa_2A_0^2/a_1)) < 0$ and $(1 - (pa_2B_0^2/a_1)) < 0$ the amplitudes A and B decrease to the same limit value. This limit value is constant and independent of the initial amplitude.
(3) for the initial amplitudes, which satisfy the relations $(1 - (pa_2A_0^2/a_1)) = 0$ and $(1 - (pa_2B_0^2/a_1)) = 0$, the motion is steady state with constant amplitudes and constant and equal frequencies $\omega_1 = \omega_2 = \sqrt{a_1b_3/pa_2}$. The motion is described as

$$z = \sqrt{\frac{a_1}{pa_2}}\left[\mathrm{cn}\left(t\sqrt{\frac{a_1b_3}{pa_2}} + \alpha_0, \frac{1}{2}\right) + \mathrm{i}\,\mathrm{cn}\left(t\sqrt{\frac{a_1b_3}{pa_2}} + \beta_0, \frac{1}{2}\right)\right]. \tag{41}$$

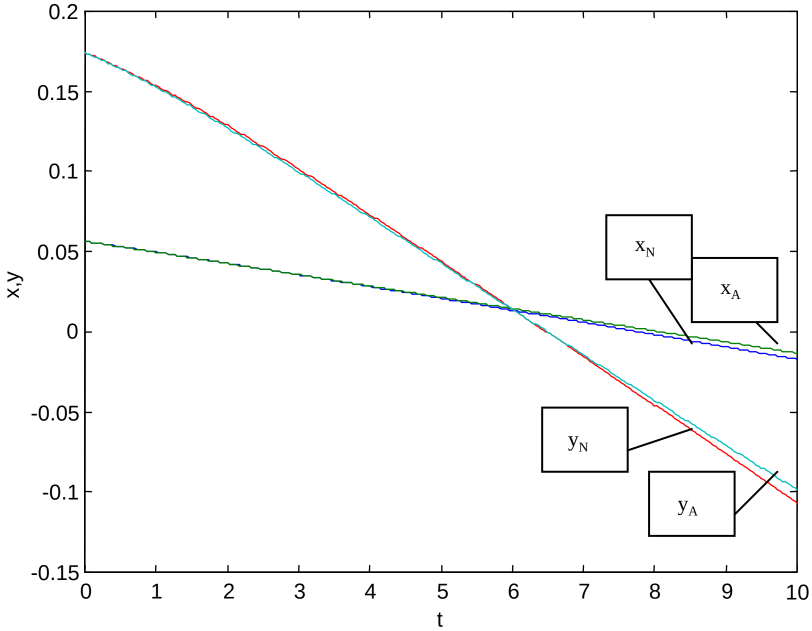

Fig. 1. The time-history diagrams obtained analytically ($x_A - t,\ y_A - t$) and numerically ($x_N - t,\ y_N - t$).

This motion is along a closed curve in $x-y$ plane. To illustrate the aforementioned result two $x-y$ curves for $\varepsilon = 0.01$, $p = 1$, and initial phase angles $\alpha_0 = \pi/3$ and $\beta_0 = \pi/6$ are plotted for initial amplitudes $A_0 = B_0 = 0.1$ when the previous condition (1) is satisfied and for $A_0 = B_0 = 0.530745$ when condition (3) is satisfied. From Fig. 2 is evident that curve (a) increases and curve (b) corresponds to the steady-state wise.

For the special case when the initial phases are the same, i.e., $\alpha_0 = \beta_0$ or $\beta_0 = \alpha_0 + 2K$, where K is the complete elliptic integral of the first kind [10], the motion is periodic and is along a line; namely

$$y = x = \pm\sqrt{\frac{a_1}{pa_2}}\,\mathrm{cn}\left(t\sqrt{\frac{a_1 b_3}{pa_2}} + \alpha_0, \frac{1}{2}\right). \tag{42}$$

(The upper sign is for the first initial phase, and the lower for the second initial condition.)

For the case when $\beta_0 = \alpha_0 + K$ or $\beta_0 = \alpha_0 + 3K$, the motion is along a symmetric closed curve

$$\frac{y^2}{(a_1/pa_2)} + \frac{x^2 - (a_1/pa_2)}{x^2 + (a_1/pa_2)} = 0. \tag{43}$$

The mathematical model (26) describes the free vibrations of a symmetric nonlinear shaft-disc system (rotor). Comparing the steady-state motion of the center of the rotor with strong nonlinearity considered in this paper and the rotor with linear properties (see [11]) it can be

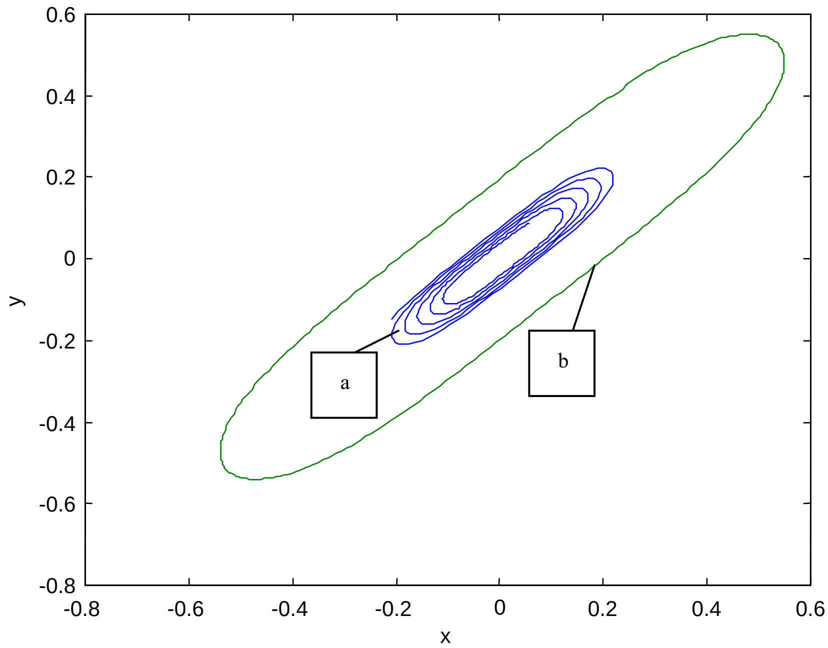

Fig. 2. The x–y diagram for (26) with $b_3 = 1$, $p = 1$, $\varepsilon = 0.01$, initial phases $\alpha_0 = \pi/3$, $\beta_0 = \pi/3$, and initial amplitudes: (a) $A_0 = B_0 = 0.1$, (b) $A_0 = B_0 = 0.530745$.

concluded that in the both cases the trajectories of the rotor center are closed curves, but their shapes differ.

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